Calcul de primitives

Pour chacune des fonctions, donner une primitive ainsi que le ou les intervalles maximaux sur lesquels elle peut être définie.

$$\boldsymbol{f(x) = \frac{1}{(x^2+1)\, Atan^3(x)}}$$

Changement de variable :

$$u = Atan(x), \qquad u' = \frac{1}{x^2+1}$$

$$f(x) = \frac{u'}{u^3} = u^{-3}\, u'$$

Primitive :

$$F(x) = \frac{u^{-2}}{-2} = -\frac{1}{2\, u^2} = -\frac{1}{2\, Atan^2(x)}$$

Intervalles maximaux : $\mathbb{R}$

$$\boldsymbol{f(x) = \frac{1}{x\, Ln(x)}}$$

Changement de variable :

$$u = Ln(x), \qquad u' = \frac{1}{x}$$

$$f(x) = \frac{u'}{u}$$

Primitive :

$$F(x) = Ln|Ln(x)|$$

Intervalles maximaux : $]0,1[,]1,+\infty[$

$$\boldsymbol{f(x) = e^x\, sin(e^x)}$$

Changement de variable :

$$u = e^x, \qquad u' = e^x$$

$$f(x) = u'\ sin(u)$$

Primitive :

$$F(x) = -cos(u) = -cos(e^x)$$

Intervalles maximaux : $\mathbb{R}$

$$\boldsymbol{f(x) = \ cos(x) \ e^x}$$

1ère méthode : double intégration par partie

$$u' = cos(x), \qquad v = e^x$$

$$u = sin(x), \qquad v' = e^x$$

Primitive :

$$F(x) = sin(x)\ e^x - \int sin(x)\ e^x$$

$$u' = sin(x), \qquad v = e^x$$

$$u = -cos(x), \qquad v' = e^x$$

$$F(x) = sin(x)\ e^x - \left(-cos(x)\ e^x + \int cos(x)\ e^x\right)$$

$$F(x) = \big(sin(x) + cos(x)\big)\ e^x - F(x)$$

$$F(x) = \frac{1}{2}\ \big(sin(x) + cos(x)\big)\ e^x$$

2ème méthode :

$$f(x) = Re\big(e^{i\,x}\ e^x\big) = Re\big(e^{(1+i)\,x}\big)$$

$$F(x) = Re\left(\frac{e^{(1+i)\,x}}{1+i}\right) = Re\left(\frac{(1-i)\ \big(cos(x) + i\ sin(x)\big)\ e^x}{(1-i)\ (1+i)}\right)$$

$$= \ Re\left(\frac{(1-i)\ \big(cos(x) + i\ sin(x)\big)\ e^x}{2}\right) = \frac{1}{2}\ \big(sin(x) + cos(x)\big)\ e^x$$

Intervalles maximaux : $\mathbb{R}$

$$\boldsymbol{f(x) = \frac{1}{sin^2(x)}}$$

Règle de Bioche :

$$f(x+\pi)\ d(x+\pi) = \frac{1}{sin^2(x+\pi)}\,dx = f(x)dx$$

Changement de variable :

$$u = tan(x), \qquad u' = \frac{1}{cos^2(x)}$$

$$f(x) = \frac{1}{sin^2(x)}\ cos^2(x)\ u' = \frac{u'}{u^2}$$

Primitive :

$$F(x) = -\frac{1}{u} = -\frac{1}{tan(x)} = -cotan(x)$$

Intervalles maximaux : $\left]-\frac{\pi}{2} + k\,\pi, \frac{\pi}{2} + k\,\pi\ \right[, k \in \mathbb{Z}$

$$\boldsymbol{f(x) = \frac{x}{tan(x^2)}}$$

Changement de variable :

$$u = x^2, \qquad u' = 2\,x$$

$$f(x) = \frac{u'}{2\,tan(u)} = \frac{1}{2}\,\frac{cos(u)}{sin(u)}u'$$

Primitive :

$$F(x) = \frac{1}{2}\,Ln|sin(u)| = \frac{1}{2}\,Ln|sin(x^2)|$$

conditions :

$$x^2 \in \left]2\,n\,\pi, \frac{\pi}{2} + 2\,n\,\pi\ \right[ou \left]\frac{\pi}{2} + 2\,n\,\pi, \pi + 2\,n\,\pi\right[, n \in \mathbb{N}$$

Intervalles maximaux :

$$\left]\sqrt{2\,n\,\pi}, \sqrt{\frac{\pi}{2} + 2\,n\,\pi}\ \right[ou \left]\sqrt{\frac{\pi}{2} + 2\,n\,\pi}, \sqrt{\pi + 2\,n\,\pi}\right[, n \in \mathbb{N}$$

$$\boldsymbol{f(x) = \frac{1}{\sqrt{\frac{5}{4} - x^2 + x}}}$$

Forme canonique :

$$\frac{5}{4} - x^2 + x = -\left(x^2 - x - \frac{5}{4}\right) = -\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{5}{4}\right) = \left(\frac{\sqrt{6}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2$$

Changement de variable :

$$u = x - \frac{1}{2}, \qquad u' = 1$$

$$f(x) = \frac{u'}{\sqrt{\left(\frac{\sqrt{6}}{2}\right)^2 - u^2}}$$

Primitive :

$$F(x) = Asin\left(\frac{2}{\sqrt{6}}\ u\right) = Asin\left(\frac{2}{\sqrt{6}}\left(x - \frac{1}{2}\right)\right)$$

Intervalles maximaux :

$$\left]\frac{1}{2} - \frac{\sqrt{6}}{2}, \frac{1}{2} + \frac{\sqrt{6}}{2}\right[$$

$$\boldsymbol{f(x) = \frac{4\ x + 2}{3\ x^2 + 7\ x + 5}}$$

Forme canonique :

$$3\ x^2 + 7\ x + 5 = 3\ \left(x^2 + \frac{7}{3}x + \frac{5}{3}\right) = 3\ \left(\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} + \frac{60}{36}\right) = 3\ \left(\left(x + \frac{7}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2\right)$$

Changement de variable :

$$u = x + \frac{7}{6}, \qquad u' = 1$$

$$f(x) = \frac{\left(4\ \left(u - \frac{7}{6}\right) + 2\right)\ u'}{3\ \left(u^2 + \left(\frac{\sqrt{11}}{6}\right)^2\right)} = \frac{2}{3}\ \frac{2\ u\ u'}{u^2 + \left(\frac{\sqrt{11}}{6}\right)^2} - \frac{8}{9}\ \frac{u'}{u^2 + \left(\frac{\sqrt{11}}{6}\right)^2}$$

Primitive :

$$F(x) = \frac{2}{3}\ Ln\left(u^2 + \left(\frac{\sqrt{11}}{6}\right)^2\right) - \frac{8}{9} \times \frac{6}{\sqrt{11}}\ Atan\left(\frac{6}{\sqrt{11}}\ u\right)$$

$$= \frac{2}{3}\ Ln\left(x^2 + \frac{7}{3}x + \frac{5}{3}\right) - \frac{16}{3\sqrt{11}}\ Atan\left(\frac{6}{\sqrt{11}}\left(x + \frac{7}{6}\right)\right)$$

Intervalles maximaux : $\mathbb{R}$

$$\boldsymbol{f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$

1ère méthode :

Changement de variable :

$$u = e^x + e^{-x}, \qquad u' = e^x - e^{-x}$$

$$f(x) = \frac{u'}{u}$$

Primitive :

$$F(x) = Ln(u) = Ln(e^x + e^{-x})$$

Intervalles maximaux : $\mathbb{R}$

2ème méthode :

$$f(x) = \frac{sh(x)}{ch(x)}$$

$$F(x) = Ln(ch(x))$$

$$\boldsymbol{f(x) = \frac{1}{x\sqrt{x-1}}}$$

Changement de variable :

$$u = \sqrt{x-1}, \qquad u^2 = x - 1, \;\; 2\,u\,u' = 1$$

$$f(x) = \frac{2\,u\,u'}{(1+u^2)\,u} = \frac{2\,u'}{1+u^2}$$

Primitive :

$$F(x) = 2\,Atan(u) = 2\,Atan\left(\sqrt{x-1}\right)$$

Intervalles maximaux : $]1,+\infty[$

$$\boldsymbol{f(x) = \frac{sin^3(x)}{cos^5(x)}}$$

Règle de Bioche :

$$f(x+\pi)\,d(x+\pi) = f(x)dx$$

Changement de variable :

$$u = tan(x), \qquad u' = \frac{1}{cos^2(x)}$$

$$f(x) = \frac{sin^3(x)}{cos^5(x)}\,cos^2(x)\,u' = \frac{sin^3(x)}{cos^3(x)}u' = u^3\,u'$$

Primitive :

$$F(x) = \frac{u^4}{4} = \frac{1}{4}tan^4(x)$$

Intervalles maximaux : $\left]-\frac{\pi}{2} + k\,\pi, \frac{\pi}{2} + k\,\pi\right[, k \in \mathbb{Z}$

$$\boldsymbol{f(x) = Ln\left(\sqrt{1-x}\right)}$$

Changement de variable :

$$t = \sqrt{1-x}, \qquad t^2 = 1 - x, \;\; 2\,t\,t' = -1$$

$$f(x) = -\,2\,t\,t'\,Ln(t)$$

Intégration par partie :

$$u' = -\,2\,t\,t', \qquad v = Ln(t)$$

$$u = -t^2, \qquad v' = \frac{1}{t}$$

Primitive :

$$F(x) = -t^2\,Ln(t) + \int t^2\,\frac{1}{t} = -t^2\,Ln(t) + \frac{1}{2}t^2$$

$$= \;(x-1)\,Ln\left(\sqrt{1-x}\right) + \frac{1}{2}\,(1-x) = \frac{1}{2}\,(1-x)\left(1 - \;Ln(1-x)\right)$$

Intervalles maximaux : $]-\infty, 1[$

$$\boldsymbol{f(x) = \frac{\sqrt{x}}{x-1}}$$

Changement de variable :

$$u = \sqrt{x}, \qquad u^2 = x,\;\; 2\,u\,u' = 1$$

$$f(x) = \frac{u}{u^2-1}\;\times 2\,u\,u' = \frac{2\,u^2\,u'}{u^2-1} = \frac{2\,(u^2-1+1)\,u'}{u^2-1}$$

$$= 2\;u' + \frac{2\;u'}{u^2-1} = 2\;u' + \frac{2\;u'}{u-1} - \frac{2\;u'}{u+1}$$

Primitive :

$$F(x) = 2\,u + 2\,Ln|u-1| - 2\,|u+1|$$

$$= 2\,\sqrt{x} + 2\,Ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|$$

Intervalles maximaux : $]0,1[\,,]1,+\infty[$

$$\boldsymbol{f(x) = \sqrt{e^x - 1}}$$

Changement de variable :

$$u = \sqrt{e^x-1}, \qquad u^2 = e^x - 1,\;\; 2\,u\,u' = e^x = u^2+1$$

$$f(x) = u\;\times \frac{2\,u\,u'}{u^2+1} = \frac{2\,u^2\,u'}{u^2+1} = \frac{2\,(u^2+1-1)\,u'}{u^2+1}$$

$$= 2\;u' - \frac{2\;u'}{u^2+1}$$

Primitive :

$$F(x) = 2\,u - 2\,Atan(u)$$
$$= 2\,\sqrt{e^x - 1} - 2\,Atan\left(\sqrt{e^x - 1}\right)$$

Intervalles maximaux : $]0, +\infty[$

$$\boldsymbol{f(x) = \sqrt{\frac{2 - x}{2 + x}}}$$

1ère méthode :

Changement de variable :

$$u = \sqrt{\frac{2 - x}{2 + x}}, \qquad u^2 = \frac{2 - x}{2 + x}, \ \ x = 2\ \frac{1 - u^2}{1 + u^2} = -2 + \frac{4}{1 + u^2},$$
$$2\,u\,u' = -\frac{4}{(2 + x)^2} = -\frac{(1 + u^2)^2}{4}$$
$$f(x) = u\ \times \frac{-8\,u\,u'}{(1 + u^2)^2} = \frac{-8\,u^2\,u'}{(1 + u^2)^2} = \frac{-8\,(u^2 + 1 - 1)\,u'}{(1 + u^2)^2}$$
$$= -\frac{8\ u'}{u^2 + 1} + \frac{8\ u'}{(u^2 + 1)^2}$$

Primitive :

$$F(x) = -8\,Atan(u) + 8\ \int \frac{u'}{(u^2 + 1)^2}$$

Changement de variable :

$$u = tan(t), \qquad u' = (1 + u^2)\,t'$$
$$\int \frac{u'}{(u^2 + 1)^2} = \int \frac{t'}{1 + tan^2(t)} = \int cos^2(t)\,t' = \int \left(\frac{1}{2} + \frac{1}{2}\ cos(2t)\right) t' = \frac{1}{2}\ t + \frac{1}{4}\ sin(2t)$$
$$= \frac{1}{2}\ Atan(u) + \frac{1}{2}\ sin(t)\,cos(t) = \frac{1}{2}\ Atan(u) + \frac{1}{2}\ \frac{sin(t)}{cos(t)}\ cos^2(t) = \frac{1}{2}\ Atan(u) + \frac{1}{2}\ \frac{u}{1 + u^2}$$
$$F(x) = -8\,Atan(u) + 4\,Atan(u) + 4\ \frac{u}{1 + u^2}$$
$$= -4\,Atan\left(\sqrt{\frac{2 - x}{2 + x}}\right) + 4\ \frac{\sqrt{\frac{2 - x}{2 + x}}}{1 + \frac{2 - x}{2 + x}} = -4\,Atan\left(\sqrt{\frac{2 - x}{2 + x}}\right) + (2 + x)\ \sqrt{\frac{2 - x}{2 + x}}$$

Intervalles maximaux : $]-2{,}2[$

2ème méthode :

Changement de variable :

$$tan(u) = \sqrt{\frac{2-x}{2+x}}, \qquad tan^2(u) = \frac{2-x}{2+x}, \qquad 1 + tan^2(u) = \frac{4}{2+x}$$

$$2\, tan(u)\left(1 + tan^2(u)\right) u' = -\frac{4}{(2+x)^2} = -\frac{\left(1+tan^2(u)\right)^2}{4}$$

$$f(x) = tan(u) \times \frac{-8\, tan(u)\, u'}{1+tan^2(u)} = \frac{-8\, tan^2(u)\, u'}{1+tan^2(u)} = \frac{-8\,(tan^2(u)+1-1)\, u'}{1+tan^2(u)}$$

$$= -8\, u' + \frac{8\ u'}{1+tan^2(u)} = -8\, u' + 8\, cos^2(u)\, u'$$

$$= -8\, u' + 8\left(\frac{1}{2} + \frac{1}{2}\, cos(2\, u)\right) u' = -4\, u' + 4\, cos(2\, u)$$

Primitive :

$$F(x) = -4\, u + 2\, sin(2\, u) = -4\, u + 4\ \frac{tan(u)}{1+tan^2(u)}$$

$$= -4\, Atan\left(\sqrt{\frac{2-x}{2+x}}\right) + 4\ \frac{\sqrt{\frac{2-x}{2+x}}}{1+\frac{2-x}{2+x}}$$

$$= -4\, Atan\left(\sqrt{\frac{2-x}{2+x}}\right) + (2+x)\ \sqrt{\frac{2-x}{2+x}}$$